Elliptic Curves Backpaper Examination

May 3 2023

This exam is of **50 marks** and is **3 hours long**. Please **read all the questions carefully**. Please feel free to use whatever theorems you have learned in class after stating them clearly. You may use the book '**The Arithmetic of Elliptic Curves' by Joseph Silverman**.

Please copy the following sentence on the first page of your answer sheet and write your name and signature.

I have not used any unfair or illegal means to answer any of the questions in this exam.

1. Consider the curve over $\mathbb{Q}$ given by

$$C : X^4 + 16Y^4 = 64Z^4$$

a. Is it a smooth curve? Prove or disprove your answer. (4)

b. Show that the map π given by

$$\pi(X, Y, Z) \to [X^4, 16Y^4, 256Z^4]$$

defines a map from C to $\mathbb{P}^1$ (3)

c. Compute the degree of the map π. (4)

d. What are the ramification points of the map? (4)

e. Use the Riemann-Hurwitz theorem to compute its genus. (5)

2. Let E and E' be elliptic curves over the finite field $\mathbb{F}_q$.

a. Show that if E is isogenous to E' then (5)

$$\#E(F_q) = \#E'(F_q)$$

b. Define the Zeta function $Z(E/\mathbb{F})q, T)$ and show that if E is isogenous to E' then (5)

$$Z(E/\mathbb{F}_q, T) = Z(E'/\mathbb{F}_q, T)$$

3 Consider the singular curve

$$C : Y^2 = X^2 + X$$

a. Count the number of points of C over $\mathbb{F}_7$. (5)

b. Compute the Zeta function of C over $\mathbb{F}_7$. (5)

4. Let $\wp$ be the Weierstrass $\wp$-function and σ be the Weierstrass σ-function assocated with an elliptic curve $E = \mathbb{C}/\Lambda$ over $\mathbb{C}$. Define the Weierstrass Zeta function by

$$\zeta(z) = \frac{1}{z} + \sum_{w\in\Lambda, w\neq 0} \left(\frac{1}{(z-w)} + \frac{1}{w} + \frac{z}{w^2} \right)$$

a. Show (3)

$$\frac{d}{dz}\log(\sigma(z)) = \zeta(z) \text{ and } \frac{d}{dz}\zeta(z) = -\wp(z)$$

b. Show that (3)

$$\zeta(-z) = -\zeta(z)$$

c. Show that for all $w \in \Lambda$ there is a constant $\eta(w)$ such that (4)

$$\zeta(z + w) = \zeta(z) + \eta(w)$$